Exploration with Macaulay2

1. **Groebner bases in various orders** In Macaulay2, create an ideal I in a polynomial ring over $\mathbb{Z}/32003$ in 6 variables generated by 5 random homogeneous quadric polynomials. For each of the monomial orders below, find the Gröbner basis, and determine the number of elements of a reduced Gröbner basis, as well as the highest degree of a Gröbner basis element.

   (a) the graded reverse lexicographic order (default in Macaulay2)

   (b) the lexicographic order (try {`Lex`}).

   (c) the elimination order, eliminating 2 variables, and breaking ties using graded reverse lexicographic order (consider `Elimination => 2`).

   (d) the product order (block order) {`3,2`}.

   Write a Macaulay2 function which, given a homogeneous ideal I, returns the pair (a, b), where a is the number of Gröbner basis elements and b is the maximum degree of a Gröbner basis element.

   As part of this problem, you should use the help system in Macaulay2, to determine how to compute Gröbner bases with respect to a specfic monomial order.

2. Let $f(x)$ be a monic polynomial in one variable, e.g., the polynomial we used in the coloring game, with zero set S (a finite number of roots).

   (a) Use Macaulay2 to find the ideal of the set in 2 variables

   $$\{(x, y) \mid x \in S, y \in S, x \neq y\}$$

   (b) Use Macaulay2 to find the ideal of the set in 3 variables

   $$\{(x, y, z) \mid x \in S, y \in S, z \in S, x \neq y, x \neq z, y \neq z\}$$

   Prove your results, with the assistance of Macaulay2.

3. Work through (and understand) the file `eg-pappus.m2` which provides a computer proof of this theorem. There are a number of ways to actually construct an ideal, some are simpler than others. Contest: who can find the simplest?

4. Look up on google (or wikipedia) the statement of the nine point circle theorem. Translate the setup and goals into ideals and polynomials, as we did for the Pappus theorem. Use this to prove the theorem.

5. **Open-ended contest!**

   Consider ideals generated by 5 quadratic polynomials. Find examples with as high regularity as you can find. What is the largest regularity that you can find?