The degree of SO(n)

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Degree of a variety

Let X be an affine algebraic variety of pure dimension d over algebraically-closed field K embedded in K^N.

Definition

The *degree* of X, $\deg X$ is the number of points in $X \cap \mathcal{L}$ where $\mathcal{L}$ is a generic codimension-d affine linear space.

$$\deg X = \#(X \cap \mathcal{L}).$$

For radical ideal $I = \mathbf{I}(X)$, say $\deg I := \deg X$.

- If $\dim X = 0$ then $\deg X = \#(X)$.
- If X is a hypersurface with $\mathbf{I}(X) = \langle f \rangle$, $\deg X = \deg f$.
- **Bézout Bound:** If X is a complete intersection of hypersurfaces $X_1, \ldots, X_r$ then $\deg X \leq \deg X_1 \cdots \deg X_r$.

Computing degree symbolically

Definition

For ideal $I \subseteq R = K[x_1, \dots, x_N]$, let $R_n \subseteq R$ denote the polynomials of degree at most n. *The Hilbert function* of R/I is $\mathrm{HF}_{R/I} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by

$$\mathrm{HF}_{R/I}(n) = \dim_K (R/I) \cap R_n.$$

The Hilbert function $\mathrm{HF}_{R/I}(n)$ is polynomial for $n >> 0$. This polynomial is the *Hilbert polynomial* of R/I, denoted $\mathrm{HP}_{R/I}(n)$.

Theorem

Suppose the Hilbert polynomial of $R/\mathbf{I}(X)$ is

$$\mathrm{HP}_{R/\mathbf{I}(X)}(n) = a_d n^d + \cdots + a_0.$$

Then

$$\dim X = d,$$

$$\deg X = d! a_d.$$

(From this fact we extend the definition of $\deg I$ to non-radical ideals and ideals over non-algebraically-closed fields.)

The Hilbert polynomial can be computed from a Gröbner basis.

Varieties O(n) and SO(n)

- O(n) is the subset of GL($\mathbb{R}^n$) preserving the standard inner product.
- SO(n) is the subset of O(n) also preserving orientation.

Both O(n) and SO(n) are algebraic groups: both *groups* and *algebraic varieties*.

$$O(n) = \{A \in \mathrm{Mat}_{n\times n} \mid A^T A = \mathrm{Id}\} \subseteq \mathbb{R}^{n^2},$$

$$a_{i,1}a_{j,1} + \cdots + a_{i,n}a_{j,n} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{for all } i \leq j.$$

The equations for SO(n) are the same but adding the degree-n equation

$$\det(A) = 1.$$

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It's convenient to work in an algebraically-closed field $\mathbb{C}$. From here, take $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ to be the Zariski closures of the above real varieties in $\mathbb{C}^{n^2}$, which does not change the degree.

Some basic facts about O(n) and SO(n)

Fact

$$\dim \mathrm{O}(n) = \dim \mathrm{SO}(n) = \frac{n(n-1)}{2}.$$

Fact

O(n) is a complete intersection of $\dfrac{n(n+1)}{2}$ quadratics.

Fact

- SO(n) is a smooth, irreducible variety.
- O(n) has two disjoint irreducible components, each isomorphic to SO(n).

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Fact

$\mathrm{O}(n)$ is a complete intersection of $\dfrac{n(n+1)}{2}$ quadratics.

Fact

- $\mathrm{SO}(n)$ is a smooth, irreducible variety.
- $\mathrm{O}(n)$ has two disjoint irreducible components, each isomorphic to $\mathrm{SO}(n)$.

Question

What is the degree of $\mathrm{SO}(n)$? $\quad$ ($\deg \mathrm{O}(n) = 2 \deg \mathrm{SO}(n)$.)

Symbolic computation of deg SO(n)

Symbolic algorithm:

n	Symbolic	H.C.	Monodromy	Formula
2	2			
3	8			
4	40			
5	384			
6	-			
7	-			
8	-			
9	-			

Limitations:

- Gröbner basis time grows badly in number of variables, which is n^2.
- We could only reach $n = 5$.
- For n even or odd we get only 2 data points each.

Computing degree numerically

Suppose X is a complete intersection, $\mathbf{I}(X) = \langle f_1, \ldots, f_r \rangle$ where $r = \operatorname{codim} X$.
Choose $\ell_1, \ldots, \ell_{N-r}$ random affine linear functionals on $\mathbb{C}^N$.

$$\deg X = \#\mathbf{V}(f_1, \ldots, f_r, \ell_1, \ldots, \ell_{N-r}).$$

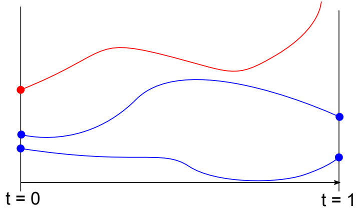
Numerical algebraic geometry can count the solutions. The *total degree homotopy* system is

$$H(t) := tF + \gamma(1-t)G$$

with

- $F = (f_1, \ldots, f_r, \ell_1, \ldots, \ell_{N-r})$,
- $G = (x_1^{d_1} - 1, \ldots, x_r^{d_r} - 1, x_{r+1} - 1, \ldots, x_N - 1)$ where $d_i = \deg f_i$ (e.g.),
- $\gamma \in \mathbb{C} \setminus \{0\}$ chosen randomly.

We know all $d_1 \cdots d_r$ solutions to $H(0) = G$. Track solutions of $H(t)$ as t goes from 0 to 1. Count how many don't go to ∞.



Numerical computation of $\deg \mathrm{SO}(n)$

Homotopy continuation algorithm:

- Recall $\mathrm{O}(n)$ is a complete intersection of $n(n+1)/2$ quadratics.
- Begin with a "start system" consisting $n(n+1)/2$ quadratics and $n(n-1)/2$ linear equations, with known solutions. E.g:

$$\begin{cases} a_{i,j}^2 - 1 & \text{for } i \le j \\ a_{i,j} & \text{for } i > j \end{cases} .$$

- Continuously deform start system to system for $\mathrm{O}(n) \cap \mathcal{L}$. Track each solution.

Limitations:

- Number of paths is $2^{n(n+1)/2}$. For $n = 6$ this is $2^{21} = 2097152$.
- We expect $\deg \mathrm{O}(6)$ to be much smaller than 2^{21}.

Mixed volume

Definition

For $f \in \mathbb{C}[x_1, \ldots, x_N]$,

$$f = c_{\alpha_1} x^{\alpha_1} + \cdots + c_{\alpha_p} x^{\alpha_p}$$

with $\alpha_1, \ldots, \alpha_p \in \mathbb{Z}^N_{\geq 0}$ and $c_{\alpha_i} \neq 0$.
The *Newton polytope* of f is $\text{conv}(\alpha_1, \ldots, \alpha_p)$.

BKK bound: For $\mathbf{I}(X) = \langle f_1, \ldots, f_N \rangle$ a complete intersection and A_i the Newton polytope of f_i

$$\#(X \cap (\mathbb{C}^*)^N) \leq \text{MV}(A_1, \ldots, A_N)$$

where MV is the mixed volume.

- The mixed volume can be much smaller than the Bézout bound.
- This suggests a more efficient homotopy start system: Polynomials with the same Newton polytopes as $(f_1, \ldots, f_N)$.
- $\text{MV}(A_1, \ldots, A_N)$ can be hard to compute, but we don't need to!
- For $\text{O}(n)$, this strategy didn't help us.

Homotopy continuation results

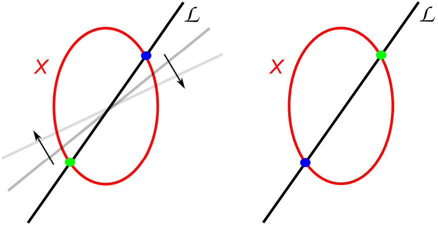
n	Symbolic	H.C.	Monodromy	Formula
2	2	2		
3	8	8		
4	40	40		
5	384	384		
6	-	-		
7	-	-		
8	-	-		
9	-	-		

Homotopy continuation computations were performed with the `NumericalAlgebraicGeometry` package for `Macaulay2` and `BERTINI`.

Numerical monodromy computation of deg SO(n)

Monodromy algorithm:

- Start with a subset of the solutions to SO(n) $\cap \mathcal{L}$ (perhaps just one point x_0).
- Moving $\mathcal{L}$ through a loop in the Grassmannian back to $\mathcal{L}$ permutes the points in SO(n) $\cap \mathcal{L}$.



- Tracking known solutions often leads to new ones.
- Repeat this process to populate all of SO(n) $\cap \mathcal{L}$.
- A solution can't leave its irreducible component, but recall SO(n) is irreducible.

Monodromy results

n	Symbolic	H.C.	Monodromy	Formula
2	2	2	2	
3	8	8	8	
4	40	40	40	
5	384	384	384	
6	-	-	4768	
7	-	-	111616	
8	-	-	-	
9	-	-	-	

Monodromy computations were performed in `Macaulay2` using the code of Duff–Hill–Jensen–Lee–Leykin–Sommars.

Kazarnovskij's formula

Theorem (Kazarnovskij)

Let G be a connected reductive group of dimension m and rank r over an algebraically closed field. If $\rho : G \to \mathrm{GL}(V)$ is a representation with finite kernel then,

$$\deg \overline{\rho(G)} = \frac{m!}{|W(G)|(e_1!e_2!\cdots e_r!)^2|\ker(\rho)|}\int_{C_V}(\check{\alpha}_1\check{\alpha}_2\cdots\check{\alpha}_l)^2 dv.$$

where $W(G)$ is the Weyl group, e_i are Coxeter exponents, C_V is the convex hull of the weights, and $\check{\alpha}_i$ are the coroots.

- representation: $\rho : \mathrm{SO}(n) \to \mathrm{GL}(\mathbb{C}^n)$ is the standard embedding.
- kernel: $\ker\rho$ is trivial.
- rank: $r = n/2$ or $(n-1)/2$ depending on n even or odd.
- dimension: $m = \binom{n}{2}$.
- size of Weyl group: $|W(\mathrm{SO}(n))| = r!2^{r-1}$ or $r!2^r$.
- Coxeter exponents: $e_1, \ldots, e_r = 1, 3, \ldots, 2r-3, r-1$ or $1, 3, \ldots, 2r-1$.
- weights: $\pm e_1, \ldots, \pm e_r$.
- coroots: $\{\check{\alpha}_1, \ldots, \check{\alpha}_l\} = \{x_i^2 \pm x_j^2\}_{1\le i<j\le r}$ or $\{x_i^2 \pm x_j^2\}_{1\le i<j\le r} \cup \{x_i^2\}_{1\le i\le r}$.

Degree formulas

Proposition (Recht–Robeva)

$$\deg \mathrm{SO}(2r) = \frac{\binom{2r}{2}!}{r!2^{r-1}(r-1)!^2 \prod_{k=1}^{r-1}(2k-1)!^2} \int_{C_V} \left(\prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \right) dv,$$

$$\deg \mathrm{SO}(2r+1) = \frac{\binom{2r+1}{2}!}{r!2^{r} \prod_{k=1}^{r}(2k-1)!^2} \int_{C_V} \left(\prod_{1\le i<j\le r} (x_i^2 - x_j^2)^2 \prod_{i=1}^{r}(2x_i)^2 \right) dv.$$

where C_V *is the cross polytope* $C_V = \mathrm{conv}(\pm e_1, \ldots, \pm e_r) \subseteq \mathbb{R}^r$*.*

Degree formulas

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where C_V is the cross polytope $C_V = \mathrm{conv}(\pm e_1,\ldots,\pm e_r)\subseteq\mathbb{R}^r$.

To evaluate these integrals:

- C_V has a standard simplices Δ_r in each orthant, and the integrand is even in each x_i.
- Rewrite the integrand as a sum of monomials with identity:

$$\prod_{1\le i<j\le r}(y_j-y_i) = \sum_{\sigma\in S_r}\mathrm{sgn}(\sigma)\prod_{i=1}^{r}y_i^{\sigma(i)-1}.$$

- Monomials integrate as $\displaystyle\int_{\Delta_r}x_1^{a_1}\cdots x_r^{a_r}\,dx = \frac{1}{(r+\sum a_i)!}\prod_{i=1}^{r}a_i!$.

Theorem

$$\deg \mathrm{SO}(n) = 2^{n-1} \det \left[\binom{2n-2i-2j}{n-2i} \right]_{1 \le i,j \le \lfloor \frac{n}{2} \rfloor}.$$

Example

$$\deg \mathrm{SO}(4) = 2^{4-1} \det \begin{bmatrix} \binom{4}{2} & \binom{2}{2} \\ \binom{2}{0} & \binom{0}{0} \end{bmatrix} = 40.$$

$$\deg \mathrm{SO}(5) = 2^{5-1} \det \begin{bmatrix} \binom{6}{3} & \binom{4}{3} \\ \binom{4}{1} & \binom{2}{1} \end{bmatrix} = 384.$$

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2	2	2	2	2
3	8	8	8	8
4	40	40	40	40
5	384	384	384	384
6	-	-	4768	4768
7	-	-	111616	111616
8	-	-	-	3433600
9	-	-	-	196968448

Real points in SO(n)

Question

How many real points can $\mathrm{SO}(n) \cap \mathcal{L}$ have (for $\mathcal{L}$ real)?

- SO(2) is a circle, so $\mathrm{SO}(n) \cap \mathcal{L}$ can have 0 or 2 real points.
- The number of real points is always even.
- The number is "usually" zero since $\mathrm{SO}(n) \cap \mathbb{R}^{n^2}$ is compact.

Taylor Brysiewicz computed the number of real points of $\mathrm{SO}(n) \cap \mathcal{L}$ many randomly chosen $\mathcal{L}$ by:

- using the monodromy algorithm to compute all solutions,
- using `alphaCertify` to determine which solutions are real.

Experimental results

Frequency of each number of points in $SO(n) \cap \mathcal{L}$:

#(Real Solutions)	$n = 3$	$n = 4$	$n = 5$
0	340141	95566	1739
2	500250	56795	776
4	655908	69501	659
6	152075	82065	633
8	17622	83635	602
10	0	64685	627
12	0	40326	653
14	0	19839	665
16	0	8499	694
18	0	2884	677
20	0	992	677
22	0	265	727
24	0	82	663
26	0	17	645
28	0	3	554
30	0	1	479
32	0	0	440
34	0	0	367
36	0	0	288
38	0	0	255
40	0	0	175
42	0	0	134
44	0	0	82
46	0	0	59
48	0	0	39
50	0	0	28
52	0	0	18
54	0	0	15
56	0	0	5
58	0	0	4
60	0	0	3
62	0	0	2
64	0	0	0
66	0	0	1